12 Quadratic Equations

Just as lines were one group of functions that have their own properties, quadratics are another. A quadratic is a function of the form

$$f(x) = ax^2 + bx + c$$

in which the highest power of x is 2. A quadratic equation is an equation that can be written as

$$ax^2 + bx + c = 0$$

In this chapter, we'll review how to solve quadratic equations. In a future chapter, we'll look at certain properties of quadratic functions and their graphs.

EXAMPLE 1:

$$x^2 + 5x - 14 = 0$$

What is one solution to the given equation?

When solving quadratic equations, factoring is the standard first approach. In this example, we can factor the left-hand side of the equation to get

$$(x + 7)(x - 2) = 0$$

For the factored expression to equal 0, either $x + 7 = 0$ or $x - 2 = 0$, so the possible solutions are $x = \boxed{-7}$ or $x = \boxed{2}$. A lesson on factoring quadratics is outside the scope of this book, so if it's something you struggle with, feel free to find an algebra textbook or an online tutorial to brush up.

In cases where a quadratic expression cannot be factored or factoring is too difficult, the quadratic formula can be used to find the solutions.

The Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Applying the quadratic formula to Example 1 yields the same answers as factoring:

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(-14)}}{2(1)} = \frac{-5 \pm \sqrt{81}}{2} = \frac{-5 \pm 9}{2} = -7 \text{ or } 2$$

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EXAMPLE 2:

$$2x^2 - 10x + 3 = 0$$

One solution to the given equation can be written as $\frac{5-\sqrt{n}}{2}$, where n is a constant. What is the value of n ?

The given quadratic expression cannot be factored, so let's use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(2)(3)}}{2(2)} = \frac{10 \pm \sqrt{76}}{4} = \frac{10 \pm 2\sqrt{19}}{4} = \frac{5 \pm \sqrt{19}}{2}$$

Therefore, one of the solutions is $\frac{5-\sqrt{19}}{2}$, and the value of n must be $\boxed{19}$.

The next few examples involve the following formulas, which you should memorize.

If $ax^2 + bx + c = 0$, then **the sum of the solutions** $= -\frac{b}{a}$ and **the product of the solutions** $= \frac{c}{a}$.

EXAMPLE 3:

$$x^2 + 5x - 14 = 0$$

What is the sum of the solutions to the given equation?

We can find the sum of the solutions without finding the solutions themselves.

$$\text{Sum of the solutions} = -\frac{b}{a} = -\frac{5}{1} = \boxed{-5}$$

Note that we already found the solutions to this equation in Example 1: -7 and 2. The sum of these solutions is indeed -5.

EXAMPLE 4:

$$2x^2 - 5x - 8 = 0$$

What is the product of the solutions to the given equation?

$$\text{Product of the solutions} = \frac{c}{a} = \frac{-8}{2} = \boxed{-4}$$

EXAMPLE 5: One of the factors of $2x^3 + 42x^2 + 216x$ is $x + c$, where c is a positive constant. What is the smallest possible value of c ?

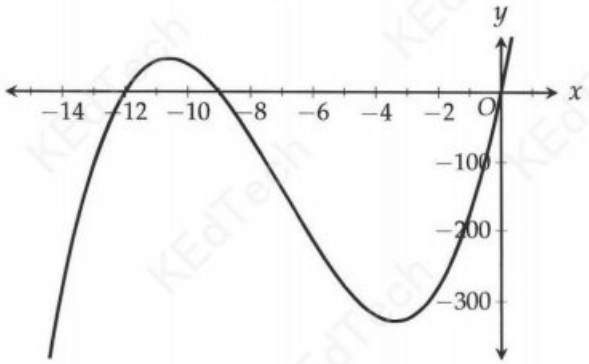
Solution 1: Factor out the greatest common factor, $2x$, from the expression and then factor the quadratic:

$$2x^3 + 42x^2 + 216x = 2x(x^2 + 21x + 108) = 2x(x + 9)(x + 12)$$

As you can see, $2x$, $x + 9$, and $x + 12$ are all factors of the expression. Since the question specifies that c is positive, c must be either 9 or 12. The smallest possible value of c is therefore $\boxed{9}$.

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Solution 2: If you have trouble with factoring, an alternative approach is to graph the given expression as a function and derive the factors from the x-intercepts of the graph. Here's the graph of $y = 2x^3 + 42x^2 + 216x$:



In the functions chapter, we learned that if the graph of a function has an x-intercept at $(n, 0)$, then $(x - n)$ must be a factor of that function. Because the graph above has x-intercepts at -12, -9, and 0, $(x + 12)$, $(x + 9)$, and x must be factors of the original expression. Therefore, the smallest possible (positive) value of c is indeed $\boxed{9}$.

Side Note: If the graph of a function has an x-intercept at 1, then $(x - 1)$ must be a factor of that function. But what if the x-intercept is at $\frac{1}{2}$ or $\frac{2}{3}$? While we could write the corresponding factors as $\left(x - \frac{1}{2}\right)$ or $\left(x - \frac{2}{3}\right)$, we typically want to avoid any fractions in these expressions. The factor $\left(x - \frac{1}{2}\right)$ is more often written as $(2x - 1)$, and $\left(x - \frac{2}{3}\right)$ is more often written as $(3x - 2)$. An easy way to tell whether two factors are equivalent is if they both equal zero for the same value of x. The factors $\left(x - \frac{1}{2}\right)$ and $(2x - 1)$ are equivalent because they both equal zero when $x = \frac{1}{2}$.

EXAMPLE 6:

$$\sqrt{x+5} + 1 = x$$

What is the solution to the given equation?

When a square root is involved, one common mistake students make is squaring each side before the square root is isolated on one side. In this example, we can't square each side right away to get rid of the square root. We first have to move the "1" from the left-hand side to the right-hand side:

$$\sqrt{x+5} = x - 1$$

Now we can square both sides.

$$(\sqrt{x+5})^2 = (x-1)^2$$
$$x + 5 = x^2 - 2x + 1$$
$$0 = x^2 - 3x - 4$$
$$0 = (x-4)(x+1)$$
$$x = -1, 4$$

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So, the solutions are -1 and 4, but hold on! We're not actually done yet. When there are square roots in the original equation, we have to check for false solutions by testing each of our values in the original equation. When $x = -1$, the left-hand side is $\sqrt{-1+5}+1 = 3$ and the right-hand side is -1. These values don't match, so -1 is not actually a solution. When $x = 4$, the left-hand side is $\sqrt{4+5}+1 = 4$ and the right-hand side is also 4. The values from both sides match so $\boxed{4}$ is the only solution.

Why do false solutions occur? Because squaring both sides has the effect of turning negative values into positive ones, which can cause a positive-negative mismatch. If we plug $x = -1$ into $\sqrt{x+5} = x-1$, we get 2 on the left but -2 on the right. One side is positive and the other is negative. Once we square both sides, however, this distinction gets lost since both sides become 4.

In summary, when you're dealing with a square root in an equation, isolate the square root before squaring each side. Once you have a set of possible solutions, check for false solutions by plugging your results into the original equation.

EXAMPLE 7:

$$(x+1)^2 + 5(x+1) - 24 = 0$$

What is the positive solution to the given equation?

To simplify the equation and make it easier to factor, we can treat $(x+1)$ as a "single entity" by substituting an arbitrary variable such as A for $x+1$. In other words, let $A = x+1$. Once we've factored the equation, we can substitute $x+1$ back in for A.

$$\begin{aligned}(x+1)^2 + 5(x+1) - 24 &= 0\\ A^2 + 5A - 24 &= 0\\ (A+8)(A-3) &= 0\\ (x+1+8)(x+1-3) &= 0\\ (x+9)(x-2) &= 0\\ x &= -9 \text{ or } 2\end{aligned}$$

Because the question asks for the positive solution, the answer is $\boxed{2}$.

EXAMPLE 8: In the xy-plane, the graph of $y = x^2 + 2x - 12$ passes through the point $(m, 6m)$, where m is a positive constant. What is the value of m ?

We're given a point on the graph, so let's plug it in.

$$\begin{aligned}y &= x^2 + 2x - 12\\ 6m &= m^2 + 2m - 12\\ 0 &= m^2 - 4m - 12\\ 0 &= (m-6)(m+2)\\ m &= 6, -2\end{aligned}$$

The question states that m is positive, so $m = \boxed{6}$.

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EXERCISE 1: Answers for this exercise start on page 356.

1

$$x^2 - 5x - 24 = 0$$

What is one of the solutions to the given equation?

2

$$x^2 - 3x - 12 = x - 7$$

What is the positive solution to the given equation?

3

$$x^2 + 7x + 19 = 28 - 5x$$

What is the sum of the solutions to the given equation?

A) -12
B) -9
C) -7
D) 12

4

$$x^2 - 10x + 7 = 0$$

Which of the following is a solution to the given equation?

A) $5 + 6\sqrt{2}$
B) $5 - 3\sqrt{2}$
C) $10 + 8\sqrt{3}$
D) $10 - 6\sqrt{3}$

5

$$-x(x + 3) = x + 4$$

What is the solution to the given equation?

6

$$x^2 + 4x + 2 = 0$$

What are the solutions to the given equation?

A) $x = -2 \pm \sqrt{2}$
B) $x = 2 \pm 2\sqrt{2}$
C) $x = -2 \pm 2\sqrt{2}$
D) $x = -4 \pm 2\sqrt{2}$

7

$$\frac{84}{x} = x - 8$$

What is the positive solution to the given equation?

8

Which of the following is a factor of $2x^2 + x - 15$?

A) $x - 3$
B) $x + 5$
C) $2x - 5$
D) $2x + 3$

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9

$$(2x-3)^2 = 4x+5$$

What is the sum of the solutions to the given equation?

10

One of the factors of $3x^3+69x^2+336x$ is $x+k$, where k is a positive constant. What is the largest possible value of k ?

11

$$x^2-12x+6=0$$

One solution to the given equation can be written as $6+\sqrt{n}$, where n is a constant. What is the value of n ?

12

$$(x+3)(x-3) = 7x-3$$

What is the product of the solutions to the given equation?

A) -12
B) -9
C) -6
D) 7

13

One of the factors of $5x^3-50x^2+120x$ is $x-m$, where m is a positive constant. What is one possible value of m ?

14

$$4x^2-(2k+1)x+k+1=0$$

In the given equation, k is a positive constant. The product of the solutions to the given equation is $w(k+1)$, where w is a constant. What is the value of w ?

A) 4
B) 2
C) $\frac{1}{2}$
D) $\frac{1}{4}$

15

$$\sqrt{(x+6)^2} = \sqrt{6-5x}$$

What is the smallest solution to the given equation?

16

$$(8x)^2-5(8x)=6$$

What is the positive solution to the given equation?

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EXERCISE 2: Answers for this exercise start on page 358.

1

$$4x^2 - 9x - 1 = 0$$

What are the solutions to the given equation?

A) $x = \dfrac{-9 \pm \sqrt{97}}{8}$

B) $x = \dfrac{-9 \pm \sqrt{65}}{8}$

C) $x = \dfrac{9 \pm \sqrt{97}}{8}$

D) $x = \dfrac{9 \pm \sqrt{65}}{8}$

2

$$-\frac{1}{2}x^2 + x = -24$$

What is the positive solution to the given equation?

3

$$x^2 + x - 5 = 0$$

One solution to the given equation can be written as $\dfrac{\sqrt{k} - 1}{2}$, where k is a constant. What is the value of k ?

4

$$2x(-x + 17) = 54 - x(5 - 3x)$$

What is the sum of the solutions to the given equation?

5

$$-3x^2 + 6x - 1 = 0$$

What is one solution to the given equation?

A) $-1 + \dfrac{\sqrt{2}}{3}$

B) $-1 + \dfrac{\sqrt{6}}{3}$

C) $1 - \dfrac{\sqrt{2}}{3}$

D) $1 - \dfrac{\sqrt{6}}{3}$

6

$$x(x - a)(x + 4) = 0$$

In the given equation, a is a positive constant. The sum of the solutions to the given equation is 6. What is the value of a ?

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7

$$x^2 - 8x = -9$$

One solution to the given equation can be written as $a - \sqrt{b}$, where a and b are constants. What is the value of $a + b$?

A) 8
B) 11
C) 18
D) 36

8

$$x^2 - 8kx - (k - 4) = 0$$

In the given equation, k is a positive constant. What is the product of the solutions to the given equation?

A) $4 - k$
B) $k - 4$
C) $-8k$
D) $8k$

9

In the xy-plane, the graph of $y = ax + 12$, where a is a positive constant, passes through the point $(-a, a)$. What is the value of a ?

10

$$2x^2 - 7x + 3 = 0$$

What is the smallest solution to the given equation?

11

Which of the following expressions is (are) a factor of $2x^2 + 11x + 12$?

I. $x + 2$
II. $2x + 1$

A) Neither I nor II
B) I only
C) II only
D) I and II

12

$$3x^2 + (108k - 51)x + 93k + 21 = 0$$

In the given equation, k is a positive constant. The sum of the solutions to the given equation is $ak + b$, where a and b are constants. What is the value of $a + b$?

13

$$72x^2 - (18p + 9q)x + 64p = -32q$$

In the given equation, p and q are positive constants. The product of the solutions to the given equation is $n(2p + q)$, where n is a constant. What is the value of n ?

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14

If $4x + k$ is a factor of $8x^3 - 26x^2 - 24x$, where k is a positive constant, what is the value of k ?

15

$$2\sqrt{4x - 19} = x - 3$$

What is the largest solution to the given equation?

16

$$(x + 11)^2 - 21(x + 11) + 54 = 0$$

What is the positive solution to the given equation?

17

$$3x^2 + 10x = 8$$

The two solutions to the given equation are a and b, and $a > b$. What is the value of b^2 ?

A) $\frac{4}{9}$

B) $\frac{2}{3}$

C) 4

D) 16

18

$$x^2 + px + 18 = 0$$

In the given equation, p is a positive constant. The solutions to the given equation are integers. Which of the following is (are) a possible value of p ?

I. 9
II. 17
III. 19

A) I only

B) II only

C) I and III only

D) I, II, and III

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